IMPLEMENTATION OF KUMAR'S CORRESPONDENCE

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Abstract. In 1997, N.M. Kumar published a paper which introduced a new tool of use in the construction of algebraic vector bundles. Given a vector bundle on projective n-space, a well known theorem of Quillen-Suslin guarantees the existence of sections which generate the bundle on the complement of a hyperplane in projective n-space. Kumar used this fact to give a correspondence between vector bundles on projective n-space and vector bundles on projective $(n-1)$-space satisfying certain conditions. He then applied this correspondence to establish the existence of many, previously unknown, rank two bundles on projective fourspace in positive characteristic. The goal of the present paper is to give an explicit homological description of Kumar's correspondence in a setting appropriate for implementation in a computer algebra system.

1. Introduction

A fundamental problem in algebraic geometry is the study, classification and construction of varieties, schemes and sheaves. These problems are related in the sense that progress in one area often leads to progress in each of the other areas. For instance, given a sheaf with interesting or unusual properties, one can often obtain correspondingly interesting varieties and schemes as degeneracy loci of the sheaf. A main focus of the present paper is an explicit homological description of a tool of use in the construction of locally free sheaves on $\mathbb{P}^n$ over an algebraically closed field, K, of arbitrary characteristic. With a slight abuse of language, we will use the term Algebraic Vector Bundle for such a sheaf. A vector bundle $\mathcal{E}$ of rank r on $\mathbb{P}^n$ is said to be of low rank if $r < n$. The co-rank of a bundle is the difference $n - r$. It appears that indecomposable low rank vector bundles on $\mathbb{P}^n$ are exceedingly rare. In fact, the only known co-rank 2 vector bundles in characteristic zero are the Horrocks-Mumford bundle on $\mathbb{P}^4$ and the Horrocks bundle on $\mathbb{P}^5$ [7, 8]. In characteristic $p > 2$ there are the additional co-rank 2 constructions of Kumar, and Kumar et al [10, 11]. In characteristic $p = 2$ there is a single example of an indecomposable co-rank 3 bundle constructed by Tango [15]. It is an open problem to construct other examples or show that they do not exist. In particular, it is unknown if there exist co-rank 2, indecomposable vector bundles on $\mathbb{P}^n$ for any value of n greater than 5. An

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interesting class of problems is concerned with establishing the existence or non-existence of higher co-rank bundles on $\mathbb{P}^n$ with prescribed properties.

The first constructions of higher co-rank algebraic vector bundles appeared in the 1970's in the papers of Horrocks-Mumford, Horrocks and Tango. After Horrock's paper in 1978, no fundamentally new, higher co-rank bundles were shown to exist for 20 years. In 1997, Kumar introduced a completely novel construction method and demonstrated its power by constructing several previously unknown co-rank 2 vector bundles in positive characteristic [10]. His method provided fuel for the additional constructions found in [11]. Kumar based his construction on the solution, by Quillen and Suslin, of the well-known Serre's conjecture on the existence of finitely generated, non-free $K[x_0, \cdots, x_n]$-modules [13, 12, 14]. For a given vector bundle on the n-dimensional projective space $\mathbb{P}^n$, the theorem of Quillen and Suslin guarantees us the existence of sections that generate the vector bundle on the complement of a hyperplane in $\mathbb{P}^n$. The pair of the vector bundle and these sections corresponds to a vector bundle on the hyperplane. Kumar gave necessary and sufficient conditions for a vector bundle on a hyperplane of $\mathbb{P}^n$ to be obtained from a vector bundle on $\mathbb{P}^n$ in this way. His correspondence between vector bundles on $\mathbb{P}^n$ and vector bundles on a hyperplane (satisfying certain conditions) were used to establish the existence of many, previously unknown, rank two vector bundles on $\mathbb{P}^4$ in positive characteristic.

The purpose of the present paper is to give an explicit homological description of Kumar's correspondence in a setting appropriate for implementation in a computer algebra system.

2. Preliminaries

2.1. **Kumar's correspondence.** Let K be a field. In 1955, J.P. Serre asked whether there exist finitely generated $K[x_0, \cdots, x_n]$-modules which are not free [13]. In 1976, Quillen and Suslin independently proved that such modules do not exist, i.e. they showed that every finitely generated projective $K[x_0, \cdots, x_n]$-module is free (cf. [12], [14]). One can apply the theorem of Quillen and Suslin to vector bundles on $\mathbb{P}^n$ as follows. Let h be a linear form in $K[x_0, \cdots, x_n]$. Let H be the hyperplane in $\mathbb{P}^n$ determined by the zeros of h. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ of rank r. By the theorem of Quillen and Suslin, $\mathcal{E}$ restricted to the complement, $\mathbb{P}^n \setminus H$ of H, is free. As a consequence, there exist r sections $s_1, \ldots, s_r \in \mathrm{H}^0\left(\mathbb{P}^n \setminus H, \mathcal{E}^\vee|_{\mathbb{P}^n \setminus H}\right)$ that generate $\mathcal{E}^\vee|_{\mathbb{P}^n \setminus H}$. It is known that for suitable integers l_i, $1 \leq i \leq r$, the sections $h^{l_i} s_i$ extend to global sections $\widetilde{s_i} \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{E}^\vee(l_i))$ (cf. [5]). Such sections define an injective morphism of sheaves $\mathcal{E} \to \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(l_i)$, which is an injective bundle map outside the divisor defined by $\widetilde{s_1} \wedge \cdots \wedge \widetilde{s_r} \in \mathrm{H}^0\left(\mathbb{P}^n, (\wedge^r \mathcal{E}^\vee)(\sum_{i=1}^r l_i)\right) \cong \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\sum_{i=1}^r l_i - c_1(\mathcal{E}))\right)$. By construction, this divisor is the m^{th} infinitesimal neighborhood H_m of H, where $m =$

$\sum_{i=1}^{r} l_i - c_1(\mathcal{E})$. In other words, there is an exact sequence

$$0 \to \mathcal{E} \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(l_i) \to \mathcal{F} \to 0, \tag{2.1}$$

where $\mathcal{F}$ is a coherent sheaf whose support is H. It is clear that the coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ possesses an $\mathcal{O}_{H_m}$-module structure, and from (2.1) it follows that the homological dimension of $\mathcal{F}$ is 1. Conversely, if there exists a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ which has an $\mathcal{O}_{H_m}$-structure, has homological dimension 1 and which allows a surjective morphism from a direct sum of r line bundles then there exists a rank r vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ and an exact sequence of type (2.1).

Let π be the finite morphism $\pi : H_m \to H$ induced by the projection $\mathbb{P}^n \setminus P \to H$ from a point $P \in \mathbb{P}^n \setminus H$. Then π_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_{H_m}$-modules to the category of quasi-coherent $\mathcal{O}_H$-modules having a $\pi_*\mathcal{O}_{H_m}$-module structure. This correspondence enables us to translate statements about quasi-coherent $\mathcal{O}_{H_m}$-modules into statements about quasi-coherent $\mathcal{O}_H$-modules.

(1) Since $\pi_*\mathcal{O}_{H_m} \simeq \bigoplus_{i=0}^{m-1} \mathcal{O}_H(-i)$, a quasi-coherent $\mathcal{O}_H$-module $\mathcal{Q}$ has a $\pi_*\mathcal{O}_{H_m}$-module structure if and only if there is a morphism $\phi : \mathcal{Q} \to \mathcal{Q}(1)$ whose m^{th} power is zero. Following Kumar, we call such a morphism a *nilpotent endomorphism of* $\mathcal{Q}$. From the theorem of Auslander and Buchsbaum it follows that a quasi-coherent $\mathcal{O}_{H_m}$-module has homological dimension 1 as a coherent sheaf on $\mathbb{P}^n$ if and only if the corresponding quasi-coherent $\mathcal{O}_H$-module has homological dimension 0, in other words, if the $\mathcal{O}_H$-module is a vector bundle.

(2) Let $\mathcal{M}$ be the direct image sheaf of $\mathcal{F}$ by π and let ϕ be the corresponding nilpotent endomorphism of $\mathcal{M}$. Since π is a finite morphism, there are natural isomorphisms $\mathrm{H}^0(\mathbb{P}^n, \mathcal{F}(-l_i)) \simeq \mathrm{H}^0(H, \mathcal{M}(-l_i))$ for all $1 \le i \le r$. We denote the restriction of $\mathcal{G} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(l_i)$ to H by $\mathcal{G}_H$. There is a surjective morphism from $\mathcal{G}$ to $\mathcal{F}$ if and only if the restriction map from $\bigoplus_{i=0}^{m-1} \mathcal{G}_H(-i)$ to $\mathcal{M}$ is surjective. The latter condition is equivalent to the condition that there exists a map $\psi : \mathcal{G}_H \to \mathcal{M}$ such that $(\phi, \psi) : \mathcal{M}(-1) \oplus \mathcal{G}_H \to \mathcal{M}$ is surjective.

Theorem 2.1. *(Kumar) There is a correspondence between* (i) *and* (ii)*:*

(i) *The set of pairs $(\mathcal{E}, s)$, where $\mathcal{E}$ is a rank r vector bundle on $\mathbb{P}^n$ and s is a morphism from $\mathcal{E}$ to $\bigoplus_{i=1}^{r} \mathcal{O}(l_i)$ with cokernel $\mathcal{F}$ satisfying:*

a) $\mathcal{F}$ is a coherent sheaf on the m^{th} infinitesimal neighborhood H_m of a hyperplane H for some positive integer m.

b) The direct image sheaf of $\mathcal{F}$ by the finite morphism $\pi : H_m \to H$ is a vector bundle $\mathcal{M}$ on H.

(ii) *The set of triples $(\mathcal{M}, \phi, \psi)$, where $\mathcal{M}$ is a vector bundle on H, $\phi : \mathcal{M} \to \mathcal{M}(1)$ is a nilpotent endomorphism and $\psi : \bigoplus_{i=1}^{r} \mathcal{O}_H(l_i) \to \mathcal{M}$*

is a morphism such that $(\phi, \psi) : \mathcal{M}(-1) \oplus \bigoplus_{i=1}^{r} \mathcal{O}_H(l_i) \to \mathcal{M}$ *is surjective.*

Proof. See [10] for a detailed proof. □

Our goal is to make explicit the procedure for computing the pair $(\mathcal{E}, s)$ corresponding to a given triple $(\mathcal{M}, \phi, \psi)$ and conversely, to make explicit the procedure for computing the triple $(\mathcal{M}, \phi, \psi)$ corresponding to a given pair $(\mathcal{E}, s)$. Let R be the homogeneous coordinate ring of $\mathbb{P}^{n-1}$ and S the homogeneous coordinate ring of $\mathbb{P}^n$. Suppose that there exists a morphism s from a rank r vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ to $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(l_i)$ satisfying the condition in Theorem 2.1. Then s induces a homomorphism from $\mathrm{H}^0_*(\mathbb{P}^n, \mathcal{E})$ to $\mathrm{H}^0_*(\mathbb{P}^n, \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(l_i)) = \bigoplus_{i=1}^{r} S(l_i)$. The sheafification of the cokernel of s is the sheaf $\mathcal{F}$. From the cokernel of s we can compute the module $F = \mathrm{H}^0_*(\mathbb{P}^n, \mathcal{F})$. Consider the R-module ${}_RF$ obtained from F by restriction of scalars. Then the sheaf associated to ${}_RF$ is $\mathcal{M}$. So the key step in each procedure is to compute the R-module ${}_RF$ from an S-module F or an S-module F from an R-module M such that ${}_RF = M$. In the following section we will discuss how to carry out these steps.

2.2. **Restriction of scalars.** Let S be the polynomial ring $K[x_0, \dots, x_n]$ and let R be the polynomial ring $K[x_0, \dots, x_{n-1}]$. For any graded S-module F we denote by ${}_RF$ the R-module obtained from F by restriction of scalars. Let Q be the quotient ring $S/(x_n^m)$ for some integer m. Suppose that F is finitely generated and has a Q-module structure (i.e. F is annihilated by the ideal (x_n^m)). Then ${}_RF$ is also finitely generated and has an ${}_RQ$-module structure. Indeed, the following proposition immediately follows from the definition of restriction of scalars.

Proposition 2.2. *Let F be a finitely generated graded S-module with minimal generating set $\mathfrak{F} = \{f_i\}_{1 \le i \le s}$. Suppose that F has a Q-module structure. Then $\mathfrak{M} = \{x_n^i f_j\}_{\substack{0 \le i \le m-1 \\ 1 \le j \le s}}$ is a generating set for ${}_RF$. Moreover the ${}_RQ$-module structure of ${}_RF$ is determined by the homomorphism $\phi : {}_RF \to ({}_RF)(1)$ defined by*

$$x_n^i f_j \mapsto \begin{cases} 0 & i \ge m-1 \\ x_n^{i+1} f_j & \textit{otherwise.} \end{cases}$$

Remark 2.3. (1) The homomorphism $\phi : {}_RF \to ({}_RF)(1)$ corresponds to multiplication $\cdot x_n : F \to F(1)$, and clearly the m^{th} power of ϕ is zero. The homomorphism $\phi : {}_RF \to ({}_RF)(1)$ obtained in this way will be called *the standard nilpotent endomorphism* of ${}_RF$.
(2) The generating set $\mathfrak{M}$ of ${}_RF$ is not always minimal. Eliminating redundant elements gives a minimal set $\mathfrak{M}' = \{g_1, \dots, g_t\}$ of generators for ${}_RF$. Let

$$M_0 \to {}_RF \to 0$$

be the corresponding epimorphism, where M_0 is a free R-module. Note that each $x_n g_i$ can be written as an R-linear combination of $g_1, \dots, g_t$:

$$x_n g_i = \sum_{j=1}^{t} a_{ij} g_j,$$

where $a_{ij} \in R$. So the matrix $(a_{ij})_{1 \le i,j \le t}$ defines a lifting $\phi_0 : M_0 \to M_0(1)$ of the standard nilpotent endomorphism ϕ of M, since ϕ sends g_i to $x_n g_i$ for $1 \le i \le t$. We call the lifting ϕ_0 of ϕ given in this way *the standard lifting of* ϕ.

A homomorphism from a finitely generated R-module M to $M(1)$ is said to be a *nilpotent endomorphism* of M if its m^{th} power is zero for some positive integer m. The functor ${}_R\cdot$ induces an equivalence of categories from the category $\mathfrak{S}_m$ of finitely generated S-modules having a $Q = S/(x_n^m)$-module structure to the category $\mathfrak{R}$ of finitely generated R-modules having an ${}_RQ$-module structure (i.e. having a nilpotent endomorphism ϕ with $\phi^m = 0$). Indeed, for an R-module $M = < g_1, \dots, g_t >$, we can define a finitely generated S-module ${}^SM$ by considering the set of all S-linear combinations of the generators of M (i.e. the set $\{b_1 g_1 + \dots + b_t g_t \mid b_i \in S\}$). Its Q-module structure is defined by

$$\phi(g_i) = x_n g_i \text{ for each } i = 1, \dots, t. \tag{2.2}$$

Obviously the functors ${}_R\cdot$ and ${}^S\cdot$ are inverse to each other.

For each $i = 1, \dots, t$, $x_n g_i$ can be written as an R-linear combination of the g_j's by (2.2), so we can define the *standard lifting* for ϕ in the same way as in Remark 2.3. The following proposition will show us how to compute from M the corresponding module ${}^SM$:

Proposition 2.4. *Let M be an object of $\mathfrak{R}$ and let ϕ be a nilpotent endomorphism of M with $\phi^m = 0$. Suppose that M has a minimal free presentation of type*

$$M_1 \xrightarrow{\alpha} M_0 \to M \to 0 \tag{2.3}$$

Then the corresponding S-module F in $\mathfrak{S}_m$ has a presentation

$$(M_1 \otimes_R S) \oplus (M_0(-1) \otimes_R S) \xrightarrow{(\alpha, \phi_0(-1) - \cdot x_n)} M_0 \otimes_R S \to F \to 0$$

where $\phi_0 : M_0 \to M_0(1)$ is the standard lifting of ϕ and $\cdot x_n$ is multiplication by x_n.

Proof. Let $\{g_1, \dots, g_t\}$ be a minimal set of generators for M. Then $F = \{b_1 g_1 + \dots + b_t g_t \mid b_i \in S\}$. Let $\phi_0(-1) = (a_{ij})_{1 \le i,j \le t}$ be the standard lifting of $\phi(-1)$. Then it follows from (2.2) that $\{g_1, \dots, g_t\}$ satisfies the relations

$$\sum_{j=1}^{t} a_{ij} g_j - x_n g_i = 0 \tag{2.4}$$

for all $i = 1, \dots, t$. So $(\alpha, \phi_0(-1) - \cdot x_n)$ forms part of a presentation matrix of F. Suppose that there is a relation on $\{g_1, \dots, g_t\}$:

$$c_1 g_1 + \cdots + c_t g_t = 0,$$

where $c_i \in S$ for each i. Without loss of generality, we may assume that each term $c_i g_i$ can be rewritten in the form $(c_i' x_n + c_i'')g_i$, where $c_i' \in S$ and $c_i'' \in R$. Let $C = (c_1, c_2, \dots, c_t)$, $C' = (c_1', c_2', \dots, c_t')$, $C'' = (c_1'', c_2'', \dots, c_t'')$, $G = (g_1, g_2, \dots, g_t)$ and $A = (a_{ij})$. By using the relations given in (2.4), we get $c_1 g_1 + \cdots + c_t g_t = CG^T = C'AG^T + C''G^T$. Set $b_j = \Sigma_{i=1}^t c_i' a_{ij} + c_j''$ and $B = [b_1, b_2, \dots, b_j]$. Then $CG^t = BG^T$. View c_i, b_i as elements of $R[x_n]$. Let $r = max\{deg(c_i) | 1 \leq i \leq t\}$ and $s = max\{deg(b_i) | 1 \leq i \leq t\}$. The construction guarantees that $s < r$. If we now repeat the same operation with $b_1 g_1 + \cdots + b_t g_t$ then in a finite number of steps we can decrease the maximum degree of the coefficients of the syzygy until all of the coefficients have degree 0, i.e. the relation becomes an R-linear combination of the g_i which is equal to 0:

$$d_1 g_1 + \cdots + d_t g_t = 0, \ d_i \in R \text{ for each } i.$$

Since we assumed that the presentation of M given in (2.3) is minimal, $(d_1, \dots, d_t)^t$ can be generated by column vectors of α. Therefore, $(\alpha, \phi_0(-1) - \cdot x_n)$ is a presentation matrix of F. $\square$

3. Algorithm

In this section we will develop a procedure for computing a rank r vector bundle on $\mathbb{P}^n$ from a given vector bundle on $\mathbb{P}^{n-1}$ satisfying the conditions in Theorem 2.1. The procedure takes as input a triple $(\mathcal{M}, \phi, \psi)$ and produces as output the corresponding pair $(\mathcal{E}, s)$. More specifically, the procedure takes as input:

- The finitely generated R-module $M = < g_1, \dots, g_t >$ with minimal free presentation
$$M_1 \xrightarrow{\alpha} M_0 \to M \to 0$$
whose associated sheaf, $\mathcal{M} = \widetilde{M}$, is locally free;
- A nilpotent endomorphism ϕ of M and its standard lifting ϕ_0;
- A homomorphism $\psi = (\psi_1, \dots, \psi_r)$ from a free module $\bigoplus_{i=1}^r R(l_i)$ to M such that the corresponding sheaf morphism from $\bigoplus_{i=1}^r \mathcal{O}(l_i)$ to $\mathcal{M}$ is a morphism such that $(\phi, \psi) : \mathcal{M}(-1) \oplus \bigoplus_{i=1}^r \mathcal{O}_H(l_i) \to \mathcal{M}$ is surjective.

The procedure produces as output:

- The finitely generated S-module E whose associated sheaf is a rank r vector bundle;
- A homomorphism $s : E \to \bigoplus_{i=1}^r S(l_i)$ such that the coherent sheaf associated to $\mathrm{Coker}(s)$ coincides with ${}^S M$.

To get the pair $(\mathcal{E}, s)$ from the triple $(\mathcal{M}, \phi, \psi)$, we take the following steps:

(i) Define a finitely generated S-module F by $\{a_1 g_1 + \cdots + a_t g_t \mid a_i \in R\}$. In practice, this module will be given as the cokernel of the homomorphism $(\alpha, \phi_0(-1) - \cdot x_n) : (M_1 \otimes_R S) \oplus (M_0(-1) \otimes_R S) \to M_0 \otimes_R S$ (see Proposition 2.4).

(ii) Define the homomorphism from $\bigoplus_{i=1}^r S(l_i)$ to F by $\psi = (\psi_1, \ldots, \psi_r)$ and compute the syzygy module $\mathrm{Syz}(\psi_1, \ldots, \psi_r)$ which represents the desired homomorphism $s : E \to \bigoplus_{i=1}^r S(l_i)$. Note that ψ_i can be written as an R-linear combination of the g_j's for each $i = 1, \ldots, t$. So a simple way of computing $\mathrm{Syz}(\psi_1, \ldots, \psi_r)$ is to determine the generating set $\{g_1, \ldots, g_t\}$ of F as a $Q = S/(x_n^m)$-module by using the presentation matrix of F given in (i). This enables us to compute $\mathrm{Syz}(\psi_1, \ldots, \psi_r)$ as a Q-module. Indeed, let N be the extension of the module $\mathrm{Syz}(\psi_1, \ldots, \psi_r)$ to S. Then $\mathrm{Syz}(\psi_1, \ldots, \psi_r)$ will be obtained as the quotient of N by $x_n^m N$.

Remark 3.1. Let (E, s) be the resulting pair. Then we want to check that $\mathcal{E} = \widetilde{E}$ is indeed a rank r vector bundle on $\mathbb{P}^n$. By construction, $\mathcal{E}$ can be regarded as a subsheaf of $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(l_i)$:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{j=1}^k \mathcal{O}_{\mathbb{P}^n}(m_j) \arrow[rr, "A"] \arrow[dr] & & \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(l_i) \arrow[r] & \cdots \\
& & \mathcal{E} \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

The entries of the j^{th} column of A define the scheme of zeros $X_{s_j} = \{s_j = 0\}$; the entries of the i^{th} row of A define the scheme of zeros $X_{\sigma_i} = \{\sigma_i = 0\}$. Recall that s is an injective bundle map outside the divisor defined by

$$x_n^m = \sigma_1 \wedge \cdots \wedge \sigma_r \in \mathrm{H}^0(\mathbb{P}^n, (\wedge^r \mathcal{E}^\vee)(\sum_{i=1}^r l_i)) \cong \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)),$$

where c_1 is the first Chern class of $\mathcal{E}$ and $m = \sum_{i=1}^r l_i - c_1$. The j^{th} column of A represents the section $t_j = s(s_j)$ of $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(l_i - m_j)$. So we have the relation of the form

$$t_{j_1} \wedge \cdots \wedge t_{j_r} = x_n^m \cdot (s_{j_1} \wedge \cdots \wedge s_{j_r}),$$

and hence we can prove that $\widetilde{E}$ is a vector bundle by checking that the ideal quotient $(I : x_n^m)$ defines the empty set in $\mathbb{P}^n$, where I is the ideal generated by the maximal minors of A.

The following examples will show how the procedure works. The procedure in the first example takes as input the twisted cotangent bundle on $\mathbb{P}^2$ and returns as output a stable rank two vector bundle on $\mathbb{P}^3$ with Chern classes $(c_1, c_2) = (0, 1)$. This bundle is the null correlation bundle on $\mathbb{P}^3$.

Example 3.2. Let $R = K[x_0, x_1, x_2]$ and let $S = K[x_0, x_1, x_2, x_3]$. Consider the following Koszul complex:

$$0 \to R(-1) \xrightarrow{\alpha_2} 3R \xrightarrow{\alpha_1} 3R(1) \xrightarrow{\alpha_0} R(2)$$

where

$$\alpha_0 = (x_0, x_1, x_2), \quad \alpha_1 = \begin{pmatrix} -x_1 & -x_2 & 0 \\ x_0 & 0 & -x_2 \\ 0 & x_0 & x_1 \end{pmatrix} \text{ and } \alpha_2 = \begin{pmatrix} x_2 \\ -x_1 \\ x_0 \end{pmatrix}.$$

Let $M = \mathrm{Im}(\alpha_1) = < s_1, s_2, s_3 >$. Then $\widetilde{M}$ is the twisted cotangent bundle $\Omega^1(2)$. The third row, t_1 of α_1, induces a map from $\Omega^1(2)$ to $\mathcal{O}(1)$ such that $t_1 \circ s_1 = 0$. So the composite of $s_1(1)$ and t_1 defines a nilpotent endomorphism ϕ of M, and hence $\widetilde{M}$. In this case, the standard lifting of ϕ is

$$\phi_0 = \begin{pmatrix} 0 & x_0 & x_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : 3R \to 3R(1).$$

This can be summarized in the following sequence of maps

$$\cdots \to R \xrightarrow{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} 3R \xrightarrow{\begin{pmatrix} -x_1 & -x_2 & 0 \\ x_0 & 0 & -x_2 \\ 0 & x_0 & x_1 \end{pmatrix}} 3R(1) \xrightarrow{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}} R(1) \xrightarrow{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} 3R(1) \to \cdots$$

The fact that $t_1 \circ s_1 = 0$ corresponds to

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x_1 & -x_2 & 0 \\ x_0 & 0 & -x_2 \\ 0 & x_0 & x_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0.$$

The map $\phi_0 : 3R \to 3R(1)$ corresponds to

$$\phi_0 = \begin{pmatrix} 0 & x_0 & x_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -x_1 & -x_2 & 0 \\ x_0 & 0 & -x_2 \\ 0 & x_0 & x_1 \end{pmatrix}.$$

By Proposition 2.4, the corresponding S-module F in $\mathfrak{S}_2$ has the following minimal presentation:

$$4S(-1) \xrightarrow{\beta_0} 3S \to F \to 0,$$

where the first column of β_0 is the presentation matrix for M (i.e. α_2) and the next three columns of β_0 are just the columns of the matrix $\phi_0(-1) - x_3 I$ where I is the 3×3 identity matrix. Thus,

$$\beta_0 = \begin{pmatrix} x_2 & -x_3 & x_0 & x_1 \\ -x_1 & 0 & -x_3 & 0 \\ x_0 & 0 & 0 & -x_3 \end{pmatrix}.$$

The other generators s_2 and s_3 of M define a homomorphism $\psi : 2R \to M$, whose lifting is given by the matrix

$$\psi_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} : 2R \to 3R.$$

This homomorphism together with the nilpotent endomorphism $\phi(-1)$ of $M(-1)$ yields a homomorphism $(\phi(-1), \psi) : M(-1) \oplus 2R \to M$. The image N is generated by the columns of the matrix

$$\begin{pmatrix} 0 & -x_0x_1 & -x_1^2 & -x_2 & 0 \\ 0 & x_0^2 & x_0x_1 & 0 & -x_2 \\ 0 & 0 & 0 & x_0 & x_1 \end{pmatrix} : 3R(-1) \oplus 2R \to 3R(1).$$

The first three columns of the matrix come from $\alpha_1\phi_0(-1)$ (i.e. multiply α_1 and ϕ_0) and the next two columns come from $\alpha_1\psi_0$ (i.e. multiply α_1 and ψ_0). The truncated modules $M_{\geq 1}$ and $N_{\geq 1}$ are isomorphic, so the map of sheaves $(\phi(-1), \psi) : \Omega^1(1) \oplus 2\mathcal{O} \to \Omega^1(2)$ is surjective. From Theorem 2.1 it follows that there exists a rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ with exact sequence

$$0 \to \mathcal{E} \to 2\mathcal{O} \to \widetilde{F} \to 0. \tag{3.1}$$

Let $\mathfrak{F} = \{f_1, f_2, f_3\}$ be the minimal generating set of F, where for each i, f_i corresponds to s_i. By construction, the surjective map from $2\mathcal{O}$ to $\widetilde{F}$ in Sequence (3.1) is induced by f_2 and f_3. Let Q be the quotient ring $S/(x_3^2)$. Then F, as a Q-module, is generated by

$$f_1 = \begin{pmatrix} 0 \\ x_1x_3 \\ x_0x_3 \end{pmatrix}, \quad f_2 = \begin{pmatrix} x_0x_3 \\ x_0x_1 + x_2x_3 \\ x_0^2 \end{pmatrix}, \quad f_3 = \begin{pmatrix} x_1x_3 \\ x_1^2 \\ x_0x_1 - x_2x_3 \end{pmatrix}$$

This can be obtained by transposing the matrix that appears in the first step of a free resolution of β_0^T over Q (i.e. find $(\mathrm{Syz}(\beta_0^T))^T$ over Q). Let F' be the module generated by f_2, f_3. The syzygy module $\mathrm{Syz}(f_2, f_3)$ over Q is generated by the columns of the matrix

$$\begin{pmatrix} -x_1x_3 & -x_1^2 & -x_0x_1 + x_2x_3 \\ x_0x_3 & x_0x_1 + x_2x_3 & x_0^2 \end{pmatrix}.$$

Let N be the extension module of F' to S. Then F' is isomorphic to N/x_3^2N, and hence over S, F' has the presentation

$$\gamma_0 = \begin{pmatrix} -x_1x_3 & -x_1^2 & -x_0x_1 + x_2x_3 & x_3^2 & 0 \\ x_0x_3 & x_0x_1 + x_2x_3 & x_0^2 & 0 & x_3^2 \end{pmatrix}.$$

This corresponds to the homomorphism $s : E \to 2S$, and hence to the injective sheaf morphism $\mathcal{E} \to 2\mathcal{O}$. Let I be the ideal generated by the 2×2 minors of γ_0. Then $(I : x_3^2)$ defines the empty set in $\mathbb{P}^3$, which implies by Remark 3.1 that $\mathcal{E}$ is a vector bundle on $\mathbb{P}^3$.

By resolving γ_0, we get a minimal free resolution of the following type for E:

$$0 \to S(-4) \to 4S(-3) \to 5S(-2) \to E \to 0 \tag{3.2}$$

From Sequence (3.2) it follows that the Chern classes of $\widetilde{E}$ are $c_1 = -2$ and $c_2 = 2$. So the corresponding normalized bundle is a stable rank two vector bundle on $\mathbb{P}^3$ with Chern classes $(c_1, c_2) = (0, 1)$.

Remark 3.3. A construction almost identical to the one outlined in the previous example can be carried out with $\Omega_{\mathbb{P}^n}(2)$ whenever n is even. The construction yields a rank n bundle on $\mathbb{P}^{n+1}$.

In the next example, we will discuss the stable rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^4$ over an algebraically closed field K of characteristic two constructed by Kumar [10]. He proved the existence of this bundle by constructing a rank three vector bundle on $\mathbb{P}^3$ over K that satisfies the conditions in Theorem 2.1. Our main goal is to describe $\mathcal{E}$ explicitly by using the algorithm.

Example 3.4. Let K be an algebraically closed field with characteristic two, let $R = K[x_0, \dots, x_3]$ and let $S = K[x_0, \dots, x_4]$. Consider the module M obtained as the cokernel of the map

$$\alpha_0 = \begin{pmatrix} 0 & 0 & x_0x_1^2 & x_1^3 \\ 0 & 0 & x_0^3 & x_0^2x_1 \\ x_2^2 & x_3^2 & 0 & 0 \\ x_0 & 0 & 0 & x_3^2 \\ 0 & x_1 & x_2^2 & 0 \\ x_1 & x_0 & x_3^2 & x_2^2 \end{pmatrix} : 2R(-4) \oplus 2R(-5) \to 3R(-2) \oplus 3R(-3).$$

Let $I_i(M)$ be the ideal of $i \times i$ minors of α_0 (i.e. a Fitting invariant of M). Then $\sqrt{I_3(M)} = (1)$ and $I_4(M) = 0$. By Fitting's Lemma, the corresponding coherent sheaf $\widetilde{M}$ is a rank three vector bundle on $\mathbb{P}^3$. Let ϕ_0 be the homomorphism from $3R(-3) \oplus 3R(-4)$ to $3R(-2) \oplus 3R(-3)$ given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1^2 & x_0^2 & x_0x_1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that ϕ_0 induces a nilpotent endomorphism ϕ of M, and hence of $\widetilde{M}$, whose third power is zero. Therefore, M corresponds to an S-module F in $\mathfrak{S}_3$. Let $\mathfrak{M} = \{g_i\}_{1 \le i \le 6}$ be a minimal generating set of M. Then F is obtained as the following set:

$$F = \{\ a_1g_1 + \cdots + a_6g_6 \mid a_i \in S \text{ for each } i = 1, \dots, 6\ \}$$

The relations among g_i's in S are, by Proposition 2.4, given by the matrix

$$(\alpha_0 |\ \phi_0 - \cdot x_4) = \begin{pmatrix} 0 & 0 & x_0x_1^2 & x_1^3 & x_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0^3 & x_0^2x_1 & 0 & x_4 & 0 & 0 & 0 & 0 \\ x_2^2 & x_3^2 & 0 & 0 & 0 & 0 & x_4 & x_1^2 & x_0^2 & x_0x_1 \\ x_0 & 0 & 0 & x_3^2 & 0 & 1 & 0 & x_4 & 0 & 0 \\ 0 & x_1 & x_2^2 & 0 & 1 & 0 & 0 & 0 & x_4 & 0 \\ x_1 & x_0 & x_3^2 & x_2^2 & 0 & 0 & 0 & 0 & 0 & x_4 \end{pmatrix}.$$

From the "ones" in this matrix, it follows that the minimal set of generators for F consists of g_1, g_2, g_3 and g_6. Eliminating the redundant elements g_4 and g_5, we obtain a minimal free presentation of F:

$$S(-3) \oplus 5S(-4) \oplus 2S(-5) \xrightarrow{\beta_0} 3S(-2) \oplus S(-3) \to F \to 0$$

where

$$\beta_0 = \begin{pmatrix} 0 & x_4^2 & 0 & 0 & 0 & x_1x_4 & x_0x_1^2 + x_2^2x_4 & x_1^3 \\ 0 & 0 & 0 & x_4^2 & x_0x_4 & 0 & x_0^3 & x_0^2x_1 + x_3^2x_4 \\ x_4 & x_0^2 & x_0x_1 & x_1^2 & x_2^2 & x_3^3 & 0 & 0 \\ 0 & 0 & x_4 & 0 & x_1 & x_0 & x_3^2 & x_2^2 \end{pmatrix}.$$

Next we define a homomorphism ψ_0 from $2R(-2)$ to $3R(-2) \oplus 3R(-3)$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

This represents a homomorphism ψ from $2R(-2)$ to M. The cokernel C of (ϕ, ψ) has the presentation matrix $(\phi_0, \psi_0, \alpha_0)$. Minimizing the generators and the corresponding relations, we obtain the following presentation matrix of C:

$$\begin{pmatrix} x_0^2 & x_0x_1 & x_1^2 & x_2^2 & x_3^2 & 0 & 0 \\ 0 & 0 & 0 & x_1 & x_0 & x_2^2 & x_3^2 \end{pmatrix}.$$

Clearly C is an R-module of finite length. From Theorem 2.1 it follows that there exist a rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^4$ and an exact sequence

$$0 \to \mathcal{E} \to 2\mathcal{O}(-2) \to \widetilde{F} \to 0. \tag{3.3}$$

By construction, the surjective map $2\mathcal{O}(-2) \to \widetilde{F}$ in Sequence (3.3) is defined by g_1 and g_2. Let Q be the quotient ring $S/(x_4^3)$. Let $P = x_0^4x_1^2 + x_0^2x_1x_3^2x_4 + x_3^4x_4^2x_0x_1^5 + x_2^2x_4(x_0^3 + x_1^3)$. Then $\mathrm{Syz}(g_1, g_2)$ is generated by the

columns of the matrix γ_0 with:

$$\gamma_0^T = \begin{pmatrix} x_0^2x_4^2 & x_1^2x_4^2 \\ x_0^4x_4 & x_0^2x_1^2x_4 + x_0x_2^2x_4^2 + x_1x_3^2x_4^2 \\ x_0^3x_1x_4 + x_0x_3^2x_4^2 & x_0x_1^3x_4 + x_1x_2^2x_4^2 \\ x_0^2x_1^2x_4 + x_0x_2^2x_4^2 + x_1x_3^2x_4^2 & x_1^4x_4 \\ x_0^3x_2^2 + x_0^2x_1x_3^2 + x_3^4x_4 & x_0x_1^2x_2^2 + x_1^3x_3^2 + x_2^4x_4 \\ x_0^4x_1^2 + x_0^3x_2^2x_4 + x_0^2x_1x_3^2x_4 & x_0^2x_1^4 + x_2^4x_4^2 \\ x_0^5x_1 + x_0^3x_3^2x_4 & x_0^3x_1^3 + x_0^2x_1x_2^2x_4 + x_2^2x_3^2x_4^2 \\ x_0^6 & x_0^3x_1^3 + x_0^2x_1x_2^2x_4 + x_2^2x_3^2x_4^2 \\ x_0^3x_1^3 + x_0x_1^2x_3^2x_4 + x_2^2x_3^2x_4^2 & P \\ x_0^2x_1^4 + x_0x_1^2x_2^2x_4 + x_1^3x_3^2x_4 + x_2^4x_4^2 & x_1^6 \end{pmatrix}.$$

Let N denote the extension module of $\mathrm{Syz}(g_1, g_2)$ to S. Since $\mathrm{Syz}(g_1, g_2)$ can be identified with N/x_4^3N, $\mathrm{Syz}(g_1, g_2)$ has, as an S-module, the minimal free presentation $\gamma = \begin{pmatrix} \gamma_0 & \gamma_1 \end{pmatrix}$, where

$$\gamma_1 = \begin{pmatrix} x_4^3 & 0 \\ 0 & x_4^3 \end{pmatrix}.$$

This corresponds to an injective sheaf morphism $s : \mathcal{E} \to 2\mathcal{O}(-2)$, whose cokernel equals $\widetilde{F}$.

Let I be the ideal generated by 2×2 minors of γ. Then the ideal quotient $(I : x_4^3)$ defines the empty subset of $\mathbb{P}^4$. By Remark 3.1, $\mathcal{E}$ is a rank two vector bundle on $\mathbb{P}^4$. The Chern classes of $\mathcal{E}$ are $c_1 = -7$ and $c_2 = 16$. These can be computed in the same way as in Example 3.2.

As a final example, we will illustrate how to determine the triple $(\mathcal{M}, \phi, \psi)$ from the pair $(\mathcal{E}, s)$. In general, this direction is easier to carry out with the main difficulty coming from producing the pair $(\mathcal{E}, s)$. We will discuss the Horrocks-Mumford bundle utilizing the ideas of Kaji to produce the sections s required in the correspondence [9].

Example 3.5. Let V be a five-dimensional vector space with basis $\{e_0, \dots, e_4\}$ over K, let W be its dual and let $\mathbb{P}^4 = \mathbb{P}(V)$ be the projective space of lines in V. The homogeneous coordinate ring $K[x_0, \dots, x_4]$ of $\mathbb{P}^4$ will be denoted by S. Consider the Koszul complex resolving $K = S/\langle W\rangle$:

$$0 \to \bigwedge^5 W \otimes S(-5) \xrightarrow{\beta_4} \cdots \xrightarrow{\beta_1} \bigwedge^1 W \otimes S(-1) \xrightarrow{\beta_0} \bigwedge^0 W \otimes S \to K \to 0$$

Recall that the i^{th} bundle of differentials $\Omega^i = \Omega^i_{\mathbb{P}^4}$ is obtained as a sheafification of the syzygy module $\mathrm{Syz}_{i+1}(K)$. By choosing appropriate bases for $\bigwedge^2 W$ and $\bigwedge^3 W$, we may suppose that $\mathrm{Syz}_3(K)$ is generated by the columns

of the following matrix:

$$\beta_2 = \begin{pmatrix} x_2 & x_3 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 & 0 \\ -x_1 & 0 & x_3 & 0 & 0 & x_4 & 0 & 0 & 0 & 0 \\ x_0 & 0 & 0 & x_3 & 0 & 0 & x_4 & 0 & 0 & 0 \\ 0 & -x_1 & -x_2 & 0 & 0 & 0 & 0 & x_4 & 0 & 0 \\ 0 & x_0 & 0 & -x_2 & 0 & 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & x_0 & x_1 & 0 & 0 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & -x_1 & -x_2 & 0 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & -x_2 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & 0 & 0 & -x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}.$$

The natural duality $\bigwedge^p V \otimes \bigwedge^p W \to K$ extends to a contraction map

$$\bigwedge^p V \otimes \bigwedge^q W \to \begin{cases} \bigwedge^{p-q} V & \text{if } p \geq q \\ \bigwedge^{q-p} W & \text{otherwise.} \end{cases}$$

Using this, the linear transformation

$$\begin{pmatrix} e_2 \wedge e_3 & e_0 \wedge e_4 & e_1 \wedge e_2 & -e_3 \wedge e_4 & e_0 \wedge e_1 \\ e_1 \wedge e_4 & e_1 \wedge e_3 & e_0 \wedge e_3 & e_0 \wedge e_2 & -e_2 \wedge e_4 \end{pmatrix}$$

from $5 \bigwedge^5 W$ to $2 \bigwedge^2 W$ induces a sheaf morphism, A, from $5 \bigwedge^5 W \otimes \mathcal{O}(-1)$ to $2\Omega^2(2)$. The matrix representation A_0 of this morphism with respect to the fixed bases for $\bigwedge^2 W$ and $\bigwedge^3 W$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.$$

Let $\beta = \begin{pmatrix} \beta_2 & 0 \\ 0 & \beta_2 \end{pmatrix}$. One can show that the ideal generated by the maximal minors of the composite of β and A_0 defines the empty set, and thus A is injective as a bundle map. Let $B_0 = A_0^T \cdot \begin{pmatrix} 0 & I_{10} \\ -I_{10} & 0 \end{pmatrix}$ where I_{10} is the 10×10 identity matrix. The matrix B_0 gives rise to a sheaf morphism B from $2\Omega^2(2)$ to $5 \bigwedge^0 W \otimes \mathcal{O}$. This sheaf morphism is surjective as a bundle map (since A is injective). A and B can be thought of as the differentials of the following complex:

$$5 \bigwedge^5 W \otimes \mathcal{O}(-1) \xrightarrow{A} 2\Omega^2(2) \xrightarrow{B} 5 \bigwedge^0 W \otimes \mathcal{O}.$$

Since A is an injective bundle map and B is a surjective bundle map the homology, $\mathcal{E} = \operatorname{Ker} B / \operatorname{Im} A$, is a rank two vector bundle on $\mathbb{P}^4$. This vector bundle is known as the Horrocks-Mumford bundle, is indecomposable and has Chern classes $c_1 = -1$ and $c_2 = 4$.

Consider the following 20×1 matrices v_1 and v_2 (discovered by Kaji [9])

$$v_1 = (0, B_2.0, 0, 0, B_6, 0, 0, 0, 0, B_{11}, B_{12}, 0, 0, B_{15}, 0, 0, B_{18}, 0, 0)^T$$

$$v_2 = \left(\; 0, C_2, C_3, 0, 0, C_6, 0, 0, 0, 0, C_{11}, C_{12}, 0, 0, 0, 0, 0, C_{18}, 0, 0 \; \right)^T$$

where

$$\begin{array}{ll}
B_2 = -x_0^5x_1 - x_0x_1^2x_2x_3x_4 - x_0^3x_3x_4^2 & C_2 = -x_0^5x_3^2 - x_0^3x_2^2x_3x_4 - x_0x_1x_2x_3^3x_4 \\
B_6 = -x_0^3x_1^2x_2 - x_0^5x_4 & C_3 = -x_0^7 \\
B_{11} = x_0^4x_1^2 + x_1^3x_2x_3x_4 + x_0^2x_1x_3x_4^2 & C_6 = -x_0^5x_2^2 - x_0^3x_1x_2x_3^2 \\
B_{12} = -x_1^3x_2^2x_4 - x_0^2x_1x_2x_4^2 & C_{11} = x_0^6x_2 + x_0^4x_1x_3^2 + x_0^2x_1x_2^2x_3x_4 \\
 & \quad\; +x_1^2x_2x_3^3x_4 \\
B_{15} = x_0^6 & C_{12} = -x_0^2x_1x_2^3x_4 - x_1^2x_2^2x_3^2x_4 \\
B_{18} = x_0^2x_1^2x_2x_4 + x_0^4x_4^2 & C_{18} = x_0^6x_3 + x_0^4x_2^2x_4 + x_0^2x_1x_2x_3^2x_4.
\end{array}$$

The matrix v_1 represents a global section s_1 of $2\Omega^2(9)$; while v_2 represents a global section s_2 of $2\Omega^2(10)$. Both v_1 and v_2 can be written as S-linear combinations of the columns of $\mathrm{Syz}(\beta \circ B_0)$, thus s_1 and s_2 correspond to global sections $\widetilde{s}_1$ and $\widetilde{s}_2$ of $\mathcal{E}(7)$ and $\mathcal{E}(8)$ respectively. Both $\widetilde{s}_1$ and $\widetilde{s}_2$ are nonzero and together generate $\mathcal{E}$ on $D_+(x_0)$. Indeed, if I is the ideal generated by the maximal minors of the matrix (v_1, v_2, A_0) then the saturation of I with respect to x_0 determines the locus of points, not on H, where s_1 and s_2 do not generate $\mathcal{E}$ (H is the hyperplane defined by $x_0 = 0$). An easy computation establishes that $V(I : (x_0)^\infty) = V((1)) = \emptyset$.

The global sections $\widetilde{s}_1$ and $\widetilde{s}_2$ can be identified with a sheaf morphism $s = (s_1, s_2)$ from $\mathcal{O}(-8) \oplus \mathcal{O}(-7)$ to $\mathcal{E}$. Recall that $\mathcal{E}^\vee$ is isomorphic to $\mathcal{E}(c_1)$ (since $\mathcal{E}$ is a rank 2 reflexive sheaf). Taking the transpose of s we obtain the following short exact sequence:

$$0 \to \mathcal{E}(-1) \xrightarrow{s^\vee} \mathcal{O}(7) \oplus \mathcal{O}(8) \to \mathcal{F} \to 0.$$

Since $\widetilde{s}_1 \wedge \widetilde{s}_2 \in \mathrm{H}^0(\mathbb{P}^4, \mathcal{E}(7) \wedge \mathcal{E}(8)) \simeq \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}(14))$ and since s_1, s_2 generate $\mathcal{E}$ away from H, the sheaf $\mathcal{F}$ can be considered as a coherent sheaf on the 14th infinitesimal neighborhood H_{14} of H. Let π be the finite morphism from H_{14} to H induced by the projection $\mathbb{P}^4 \setminus P \to H$ from a point P off H. Then the direct image sheaf of $\mathcal{F}$ by π is a rank fourteen vector bundle on $H \simeq \mathbb{P}^3$. We denote this bundle by $\mathcal{M}$.

Let R be the quotient ring $S/(x_0)$ and let F be the graded T-module $\mathrm{H}^0_*\mathcal{F}$. Then the graded R-module $M = \mathrm{H}^0_*\mathcal{M}$ is the graded R-module ${}_RF$ obtained from F by restriction of scalars. It is straightforward to determine that F has a minimal free presentation of the following form:

$$15S \xrightarrow{P} S(8) \oplus S(7) \oplus 5S(1) \to F \to 0,$$

Let $\mathfrak{F} = \{f_i\}_{1 \le i \le 7}$ be the minimal generating set of F. Then it follows from Proposition 2.2 that $\mathfrak{M} = \left\{ \; x_0^i f_j \mid 0 \le i \le 13, 1 \le j \le 7 \; \right\}$ is a set of generators for M. The relations among these generators of M can be derived from the presentation matrix P of F. Let $P[:, k]$ be the k^{th} column of P and let Q be the presentation matrix of M with respect to $\mathfrak{M}$. For each

$1 \le k \le 15$, we have a syzygy of the form

$$\sum_{i=1}^{7} P[i,k] f_i = 0.$$

Then, since

$$P[i,k] = \sum_{t=0}^{13} Q[7t+i,k] x_0^t$$

we can obtain the entries of $Q[:,k]$ from the entries of $P[:,k]$.

Choosing appropriate bases for F_0 and F_1, one can explicitly write P. For example, the first column of P is

$$P[:,1] = \begin{pmatrix} P[1,1] & P[2,1] & x_3 & 0 & 0 & 0 & 0 \end{pmatrix}^T,$$

where

$$\begin{aligned} P[1,1] &= x_0^6 x_2^2 - x_1^3 x_2^4 x_3 + 2x_0^2 x_1 x_2^3 x_3 x_4 + x_0 x_1^4 x_3^2 x_4 - 3x_1^2 x_2^2 x_3^3 x_4 \\ &\quad - x_0^2 x_2 x_3^3 x_4^2 + x_1 x_3^5 x_4^2 - x_0 x_1^2 x_2 x_3 x_4^3 - x_2^3 x_3^2 x_4^3 + x_0^3 x_3 x_4^4, \\ P[2,1] &= x_0^4 x_1^2 x_2 - x_0^3 x_2^3 x_3 + x_0 x_1 x_2^2 x_3^3 + x_0^6 x_4 + x_1^3 x_2^2 x_3 x_4 - x_0 x_3^5 x_4 \\ &\quad + 2x_0^2 x_1 x_2 x_3 x_4^2 + x_1^2 x_3^3 x_4^2 - x_2 x_3^2 x_4^4. \end{aligned}$$

We have $P[1,1] = Q[1,1] + Q[8,1]x_0 + Q[15,1]x_0^2 + Q[22,1]x_0^3 + Q[36,1]x_0^6$, where

$$\begin{aligned} Q[1,1] &= -x_1^3 x_2^4 x_3 - 3x_1^2 x_2^2 x_3^3 x_4 + x_1 x_3^5 x_4^2 - x_2^3 x_3^2 x_4^3 \\ Q[8,1] &= x_1^4 x_3^2 x_4 - x_1^2 x_2 x_3 x_4^3 \\ Q[15,1] &= 2x_1 x_2^3 x_3 x_4 - x_2 x_3^3 x_4^2 \\ Q[22,1] &= x_3 x_4^4 \\ Q[36,1] &= x_2^2 \end{aligned}$$

Likewise,

$$P[2,1] = Q[2,1] + Q[9,1]x_0 + Q[16,1]x_0^2 + Q[23,1]x_0^3 + Q[30,1]x_0^4 + Q[37,1]x_0^6$$

where

$$\begin{aligned} Q[2,1] &= x_1^3 x_2^2 x_3 x_4 + x_1^2 x_3^3 x_4^2 - x_2 x_3^2 x_4^4 \\ Q[9,1] &= x_1 x_2^2 x_3^3 - x_3^5 x_4 \\ Q[16,1] &= 2x_1 x_2 x_3 x_4^2 \\ Q[23,1] &= -x_2^3 x_3 \\ Q[30,1] &= x_1^2 x_2 \\ Q[37,1] &= x_4. \end{aligned}$$

Finally, $Q[3,1] = x_3$ is the remaining nonzero entry in $Q[:,1]$ (since $P[i,1] = 0$ for $4 \le i \le 7$).

Working our way through the other columns of P, the entire matrix Q can be obtained (and has $98 = 14 \cdot 7$ rows and 15 columns). Upon obtaining Q,

one finds that $Q[12,6], Q[10,8], Q[11,10], Q[13,14], Q[14,15], Q[51,12]$ and $Q[57,13]$ are the only entries of Q which are constant and nonzero. Furthermore, each of $\{x_0^i f_j | i \geq 1, j \geq 3\}$, $\{x_0^i f_1 | i \geq 8\}$ and $\{x_0^i f_2 | i \geq 7\}$, can be written as R-linear combinations of

$$\mathbf{G} = \{f_1, f_2, \dots, f_7\} \cup \{x_0^i f_1 | 1 \leq i \leq 7\} \cup \{x_0^i f_2 | 1 \leq i \leq 6\}.$$

These linear combinations give rise to the standard nilpotent endomorphism of M. Let g_j denote the j^{th} entry of $\mathbf{G}$ for $1 \leq j \leq 20$ and let

$$M_0 \xrightarrow{\mathbf{G}} M \to 0$$

be the map associated to the minimal set of generators of M. Each $x_0 g_i$ can be written as an R-linear combination of $g_1, \dots, g_{20}$:

$$x_0 g_i = \sum_{j=1}^{20} a_{ij} g_j.$$

The matrix $(a_{ij})_{1 \leq i,j \leq 20}$ is the standard lifting of the standard nilpotent endomorphism ϕ of M (see Remark 2.3). By construction, the first two generators g_1 and g_2 of M form a homomorphism ψ from $R(7) \oplus R(8)$ to M such that the cokernel of $(\phi[-1], \psi) : M(-1) \oplus R(7) \oplus R(8) \to M$ is a finite-length R-module.

It is interesting to note that the rank fourteen vector bundle $\mathcal{M}$ can be written as the direct sum of nine line bundles and an indecomposable rank five vector bundle.

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